 P-ISSN: 2502-6526 E-ISSN: 2656-0615

# SUPER GRACEFUL LABELING FOR A CLASS OF TREES

**Purwanto and Ariska Puji Rahayu**
Department of Mathematics, Universitas Negeri Malang, Malang, Indonesia
Email : purwanto.fmipa@um.ac.id, ariska.pujirahayu@gmail.com

***Abstrak***

*Let $G$ be a graph having vertex set $V(G)$, edge set $E(G)$, number of vertices $|V(G)| = p$, and number of edges $|E(G)| = q$. A super graceful labeling is a bijection function $f: V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, p + q\}$ such that $f(uv) = |f(u) - f(v)|$ for every $uv \in E(G)$. A graph is called a super graceful graph if it admits a super graceful labeling. A tree $K_{1,m(n)}$ is a tree obtained from a star on $m + 1$ vertices by joining each vertex (one vertex if $m = 1$) of degree 1 , $v_i$, $i = 1, 2, \dots, m$, to different $n$ new vertices $v_{i,1}, v_{i,2}, \dots, v_{i,n}$. Super gracefulness of many classes of graphs, including special classes of trees, has been studied in the literature. In this paper we present a super graceful labeling of the tree $K_{1,m(n)}$.*

***Key Word:*** *Super Graceful; Class of Trees*

## 1. INTRODUCTION

For our purposes, all graphs are finite, loopless and have no multiple edges. For most part of our notation and terminology we follow that of Diestel (R. Diestel, 200). We let $G$ be a graph having vertex set $V(G)$, edge set $E(G)$, number of vertices $|V(G)| = p$, and number of edges $|E(G)| = q$.

A graph labeling is a function from the set of vertices or the set of edges, or both, subject to a certain condition. A lot of graph labelings have been studied in the literature, see (J. A. Gallian, 2018). We follow (M. A. Perumal , 2011: 382-404) for the following definition of super graceful labeling. Let $G$ be a graph. A super graceful labeling is a bijection function $f: V(G) \cup E(G) \rightarrow \{1, 2, 3, \dots, p + q\}$ such that $f(uv) = |f(u) - f(v)|$ for every $uv \in E(G)$. A graph is called a super graceful graph if it admits a super graceful labeling. In (N. Hartsfield and G. Ringel) this labeling is called consecutive labeling.

A tree is a connected graph with no cycles, and a star is tree obtained from a vertex joined to a number of vertices. A tree $K_{1,m(n)}$ is a tree obtained from a star on $m + 1$ vertices by joining each vertex (one vertex, if $m = 1$) of degree 1, $v_i$, $i = 1, 2, \dots, m$, to different $n$ new vertices $v_{i,1}, v_{i,2}, \dots, v_{i,n}$. A tree $K_{1,3(4)}$ is as in Figure 1.

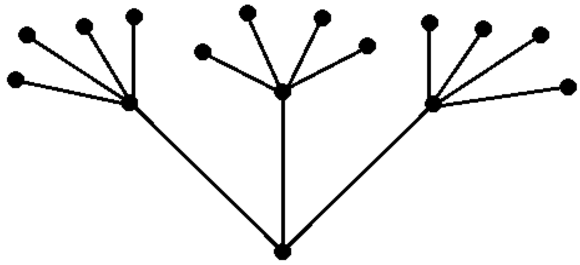

**Figure 1. Tree $K_{1,3(4)}$**

Super gracefulness of many classes of graphs, including special classes of trees, has been studied in the literature. For example, Perumal et al (M. A. Perumal , 2011: 382-404) studied the super gracefulness for cycles, paths, and regular caterpillar. Further, Lau et al (G. C. Lau, W. C. Shiu, 2016 : 200-209) studied the super gracefulness for special tripartite graphs, union of stars, and some families of trees. In this paper we present a super graceful labeling a class of trees, $K_{1,m(n)}$.

## 2. RESULTS OF THE STUDY

Perumal et al (M. A. Perumal , 2011: 382-404) define a graph $S_{m,n}$ as a graph formed from $m$ paths $u_1^0, u_1^1, \dots, u_1^n, u_2^0, u_2^1, \dots,\ u_2^n, \dots, u_m^0, u_m^1, \dots, u_m^n$ and identity the vertices $u_1^0, u_2^0, \dots, u_m^0$ with $u_0$, and show that the graph $S_{m,n}$ in super graceful. When $n = 1$, we have $S_{m,n} = K_{1,m(1)}$, and this implies that $K_{1,m(1)}$ is super graceful. In this paper we present a super graceful labeling of the tree $K_{1,m(n)}$.

By the definition, the number of vertices and the number of edges of $K_{1,m(n)}$ are $p = |V(K_{1,m(n)})| = mn + m + 1$ and $q = |E(K_{1,m(n)})| = mn + m$, respectively, and $p + q = 2mn + 2m + 1$. The following Theorem 2.1 shows an easy way to label $K_{1,m(n)}$ with a super graceful labeling. We are concerned about how to do the labeling.

**Theorem 2.1.** Let $m$ and $n$ be positif integers and $K_{1,m(n)}$ be a tree with the vertex set
$V(K_{1,m(n)}) = \{v_0, v_1, v_2, \dots, v_{m,} v_{1,1}, v_{1,2}, \dots, v_{1,n}, v_{2,1}, v_{2,2}, \dots, v_{2,n}, \dots, v_{m,1}, v_{m,2}, \dots, v_{m,n}\}$,
and the edge set
$E(K_{1,m(n)}) =$
$\{v_0v_1, v_0v_2, \dots v_0v_m, v_1v_{1,1}, v_1v_{1,2}, \dots, v_1v_{1,n}, v_2v_{2,1}, v_2v_{2,2}, \dots, v_2v_{2,n}, \dots, v_mv_{m,1},$
$v_2v_{2,n}, \dots, v_mv_{m,1}, v_mv_{m,2}, \dots, v_mv_{m,n}\}$. Then the labeling:

$f: V(G) \cup E(G) \to \{1, 2, 3, \dots, 2mn + 2m + 1\}$, where

$f(v_0) = 2(n + 1) - 1,$

$f(v_1) = 2m(n + 1) + 1,$

$f(v_i) = 2m(n + 1) + 4(n + 1) - 2(n + 1)i - 1, \quad \text{for } 2 \le i \le m,$

$f(v_{i,j}) = -2(n + 1) + 2(n + 1)i + 2j - 1, \quad \text{for } 1 \le i \le m \text{ and } 1 \le j \le n,$

and

$f(v_o v_1) = |2m(n+1) - 2n|,$
$f(v_0 v_i) = |2m(n+1) - 2(i-1)(n+1)|,$ for $2 \le i \le m,$
$f(v_1 v_{1,j}) = |2m(n+1) + 2 - 2j|,$ for $1 \le j \le n$
$f(v_i v_{i,j}) = |2m(n+1) + 2(3-2i)(n+1) - 2j|,$ for $2 \le i \le m$ and $1 \le j \le n,$

is a super graceful labeling.

**Proof.** It is obvious that

$$p = \left|V\left(K_{1,m(n)}\right)\right| = mn + m + 1,$$
$$q = \left|E\left(K_{1,m(n)}\right)\right| = mn + m,$$

and so

$$p + q = 2mn + 2m + 1.$$

Further, $f(v)$ is odd for every $v \in V(K_{1,m(n)})$, $f(uv)$ is even for every $uv \in E(K_{1,m(n)})$, and $f(uv) = |f(u) - f(v)|$.

Let $2 \le i \le m$. Then

$$\begin{aligned} f(v_i) &= 2m(n+1) + 4(n+1) - 2(n+1)i - 1 \\ &\ge 2m(n+1) + 4(n+1) - 2(n+1)m - 1 \\ &= 4n + 3, \end{aligned}$$

and

$$\begin{aligned} f(v_i) &= 2m(n+1) + 4(n+1) - 2(n+1)i - 1 \\ &\le 2m(n+1) + 4(n+1) - 2(n+1)2 - 1 \\ &= 2mn + 2m - 1. \end{aligned}$$

These imply that, for every $1 \le i \le m$,

$$1 \le f(v_i) \le 2mn + 2m + 1 = p + q.$$

Now let $1 \le i \le m$ and $1 \le j \le n$. We have

$$\begin{aligned} f\left(v_{i,j}\right) &= -2(n+1) + 2(n+1)i + 2j - 1 \\ &\ge -2n + 2(n+1) + 2 - 3 \\ &\ge 1, \end{aligned}$$

and

$$\begin{aligned} f\left(v_{i,j}\right) &= -2n + 2(n+1)i + 2j - 3 \\ &\le -2n + 2(n+1)m + 2n - 3 \\ &= 2mn + 2m - 3. \end{aligned}$$

These imply that

$$1 \le f\left(v_{i,j}\right) \le 2mn + 2m + 1 = p + q.$$

Thus we find that, for every $v \in V(K_{1,m(n)})$,

**3.** $1 \le f(v) \le p + q.$

Since, for every $uv \in E(K_{1,m(n)})$, $f(uv) = |f(u) - f(v)|$, then

$$1 \le f(uv) \le p + q - 1 = 2mn + 2m.$$

Thus, we have proved that $f(v), f(uv) \in \{1, 2, 3, \dots, p+q\}$.

We will show if $u \ne v$, then $f(u) \ne f(v)$. It is easy to see that, for $0 \le r, s, \le m$, if $r \ne s$ then $f(v_r) \ne f(v_s)$.

Let $1 \leq i, r \leq m$ and $1 \leq j, s \leq n$. For fixed $i$ and different, if $j \neq s$ then $f(v_i v_{i,j}) \neq f(v_i v_{i,s})$; the maximum value of $f(v_{i,j}) = -2n + 2(n+1)i + 2j - 3$ is

$$f(v_{i,j}) = -2n + 2(n+1)i + 2n - 3$$

and the minimum values of $f(v_{i+1,j}) = -2n + 2(n+1)(i+1) + 2j - 3$ is

$$\begin{aligned} f(v_{i+1,j}) &= -2n + 2(n+1)(i+1) + 2 - 3 \\ &= -2n + 2(n+1)i + 2n + 1 \\ &> f(v_{i,j)}). \end{aligned}$$

Thus, if $(i,j) \neq (r,s)$ then $f(v_{i,j}) \neq f(v_{r,s})$.
It is obvious that

$$f(v_0) \neq f(v_1).$$

Let $2 \leq i \leq m$. Then

$$\begin{aligned} f(v_i) - f(v_0) &= (2m(n+1) + 4(n+1) - 2(n+1)i - 1) - (2(n+1) - 1) \\ &= 2m(n+1) + 2(n+1) - 2(n+1)i \\ &\geq 2m(n+1) + 2(n+1) - 2(n+1)m \\ &= 2n + 2, \end{aligned}$$

and so

$$f(v_0) \neq f(v_i).$$

Let $1 \leq j \leq n$. Then

$$\begin{aligned} f(v_0) - f(v_{1,j}) &= (2(n+1) - 1) - (-2n + 2(n+1)1 + 2j - 3) \\ &= 2n - 2j + 2 \\ &\geq 2, \end{aligned}$$

and so

$$f(v_0) \neq f(v_{1,j}).$$

Let $2 \leq i \leq m$. Then

$$\begin{aligned} f(v_{i,j}) - f(v_0) &= (-2n + 2(n+1)i + 2j - 3) - (2(n+1) - 1) \\ &\geq (-2n + 2(n+1)2 + 2j - 3) - (2(n+1) - 1) \\ &\geq 2j, \end{aligned}$$

and so

$$f(v_0) \neq f(v_{i,j}).$$

Let $2 \leq i \leq m$. Then

$$\begin{aligned} f(v_1) &= 2m(n+1) + 1 \\ &> 2m(n+1) + 4(n+1) - 2(n+1)2 - 1 \\ &\geq 2m(n+1) + 4(n+1) - 2(n+1)i - 1 \\ &= f(v_i), \end{aligned}$$

and so

$$f(v_1) \neq f(v_i).$$

Let $1 \leq i \leq m$ and $1 \leq j \leq n$. Then

$$\begin{aligned} f(v_1) &= 2m(n+1) + 1 \\ &> -2n + 2(n+1)m + 2n - 3 \\ &\geq -2n + 2(n+1)i + 4j - 3 \\ &= f(v_{i,j}), \end{aligned}$$

and so

$$f(v_1) \neq f(v_{i,j}).$$

Now we will show that for any integers $r$, $s$, and $j$, $2 \leq r \leq m$, $1 \leq s \leq m$, and $1 \leq j \leq n$, $f(v_r) \neq f(v_{r,j})$. Suppose $f(v_r) = f(v_{s,j})$. Then $2m(n+1) + 4(n+1) - 2(n+1)r - 1 = -2n + 2(n+1)s + 2j - 3$, $2m(n+1) + 4(n+1) - 2(n+1)r - 1 = -2(n+1) + 2(n+1)s + 2j - 1$, $2m(n+1) + 6(n+1) - 2(n+1)r - 2(n+1)s = 2j$, $(m + 3 - r - s)(n+1) = j$.

It is impossible since $1 \leq j \leq n$. Thus

$$f(v_r) \neq f(v_{s,j}).$$

This complete the proof that if $u \neq v$, then $f(u) \neq f(v)$.

We will show that if $uv \neq wx$, then $f(uv) \neq f(wx)$. Note that

$f(v_o v_1) = |2m(n+1) - 2n| = 2m(n+1) - 2n$,

$f(v_0 v_i) = |2m(n+1) - 2(i-1)(n+1)|$

$= 2m(n+1) - 2(i-1)(n+1)$, for $2 \leq i \leq m$,

$f(v_1 v_{1,j}) = |2m(n+1) + 2 - 2j|$

$= 2m(n+1) + 2 - 2j$, for $1 \leq j \leq n$

$f(v_i v_{i,j}) = |2m(n+1) + 2(3-2i)(n+1) - 2j|$

$$= \begin{cases} 2m(n+1) + 2(3-2i)(n+1) - 2j, & \text{for } 2 \leq i \leq \frac{m}{2} + 1 \text{ and } 1 \leq j \leq n \\ -2m(n+1) - 2(3-2i)(n+1) + 2j, & \text{for } \frac{\mathrm{m}}{2} + \frac{3}{2} \leq i \leq m \text{ and } 1 \leq j \leq n \end{cases}$$

It is easy to see that, for $2 \leq i \leq m$,

$$(v_o v_1) \neq f(v_0 v_i),$$

and for $1 \leq j \leq n$

$$f(v_o v_1) \neq f(v_1 v_{1,j}).$$

Suppose $f(v_o v_1) = f(v_i v_{i,j})$, for $2 \leq i \leq \frac{m}{2} + 1$ and $1 \leq j \leq n$. Then

$$\begin{aligned} 2m(n+1) - 2n &= 2m(n+1) + 2(3-2i)(n+1) - 2j, \\ -2n &= 2(3-2i)(n+1) - 2j \\ &\leq 2(3-2(2))(n+1) - 2j, \\ -2n &\leq -2(n+1) - 2j, \end{aligned}$$

a contradiction. Suppose $f(v_o v_1) = f(v_i v_{i,j})$, for $\frac{\mathrm{m}}{2} + \frac{3}{2} \leq i \leq m$ and $1 \leq j \leq n$ . Then

$$\begin{aligned} 2m(n+1) - 2n &= -2m(n+1) - 2(3-2i)(n+1) + 2j, \\ 2m(n+1) - 2n &= -2m(n+1) + 2(2i-3)(n+1) + 2j \\ 2m(n+1) - 2n &\leq -2m(n+1) + 2(2m-3)(n+1) + 2j \\ -2n &\leq -6(n+1) - 2j, \end{aligned}$$

a contradiction. Thus we find

$$f(v_o v_1) \neq f(v_i v_{i,j}).$$

Let $2 \leq i \leq m$ and $1 \leq j \leq n$. Suppose $f(v_0 v_i) = f(v_1 v_{1,j})$. Then

$$\begin{aligned} 2m(n+1) - 2(i-1)(n+1) &= 2m(n+1) + 2 - 2j, \\ (i-1)(n+1) &= (j-1), \end{aligned}$$

a contradiction since $i \geq 2$ and $1 \leq j \leq n$. Thus

$$f(v_0 v_i) \neq f(v_1 v_{1,j}).$$

Let $r$, $s$, and $j$ be integers, $2 \leq r \leq m$, $1 \leq s \leq m$, and $1 \leq j \leq n$. Suppose $f(v_0 v_r) = f(v_s v_{s,j})$. Then, for $2 \leq s \leq \frac{m}{2} + 1$,

$$2m(n+1) - 2(r-1)(n+1) = 2m(n+1) + 2(3-2s)(n+1) - 2j,$$
$$-2(r-1)(n+1) = 2(3-2s)(n+1) - 2j,$$
$$j = (r+2-2s)(n-1),$$

A contradiction since $1 \leq j \leq n$. Similarly, for $\frac{\mathrm{m}}{2} + \frac{3}{2} \leq s \leq m$,

$$2m(n+1) - 2(r-1)(n+1) = -2m(n+1) - 2(3-2s)(n+1) + 2j,$$
$$j = (2m - r + 4 - 2s)(n+1),$$

a contradiction since $1 \leq j \leq n$. Thus

$$f(v_0 v_r) \neq f(v_s v_{s,j}).$$

Let $r$, $s$, and $j$ be integers, $2 \leq s \leq m$ and $1 \leq r, j \leq n$. Suppose $f(v_1 v_{1,r}) = f(v_s v_{s,j})$. Then, for $2 \leq s \leq \frac{m}{2} + 1$ and $1 \leq j \leq n$,

$$2m(n+1) + 2 - 2r = 2m(n+1) + 2(3-2s)(n+1) - 2j,$$
$$1 - r = (3-2s)(n+1) - j$$
$$\leq (3 - 2(2))(n+1) - j,$$
$$1 - r \leq -(n+1) - j,$$

a contradiction since $1 \leq r, j \leq n$. Similarly, for $\frac{\mathrm{m}}{2} + \frac{3}{2} \leq s \leq m$,

$$2m(n+1) + 2 - 2r = -2m(n+1) - 2(3-2s)(n+1) + 2j,$$
$$1 - r = (-2m - 3 + 2s)(n+1) + j$$
$$\geq (-2m - 3 + (m+3))(n+1) + j$$
$$= -m(n+1) + j,$$

A contradiction since $m \geq 2$. Thus

$$f(v_1 v_{1,r}) \neq f(v_s v_{s,j}).$$

This complete the proof that if $uv \neq wx$, then $f(uv) \neq f(wx)$.

For the function $f$ we have seen that $f(v)$ is odd for every $v \in V(K_{1,m(n)})$, $f(uv)$ is even for every $uv \in E(K_{1,m(n)})$, if $u \neq v$ then $f(u) \neq f(v)$, and if $uv \neq wx$ then $f(uv) \neq f(wx)$. Further $|V(G) \cup E(G)| = |\{1, 2, 3, \ldots, p + q\}|$. We can conclude that the function $f$ is bijective. Furthermore, since $f(uv) = |f(u) - f(v)|$ for every $u, v \in V(K_{1,m(n)})$, then $f$ is super graceful labeling.

Figure 2. is an example of super graceful labeling of $K_{1,3(4)}$.

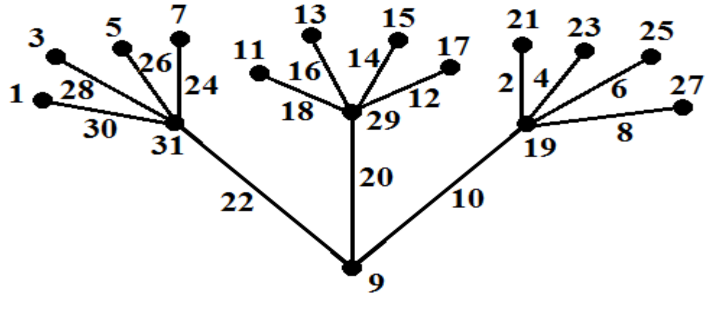

**Figure 2. Super graceful labeling of $K_{1,3(4)}$**

## 4. Conclutions

In Theorem 2.1 we label graph $K_{1,m(n)}$ by a graceful labeling. There are many problems concerning graceful labeling that have not been solved. One of the problems is as follows. Let $K_{1,l(m(n))}$ be a graph obtained from $K_{1,l(m)}$ by joining each vertex of degree 1 to different $n$ new vertices. One can try to do the following exercise: Find out whether the graph $K_{1,l(m(n))}$ is graceful labeling or not.

## 5. References


R. Diestel, 2000, *Graph Theory*.

J. A. Gallian, 2018, “A Dynamic Survey of Graph Labeling,”

M. A. Perumal, S. Navaneethakrishnan, S. Arockiaraj, and A. Nagarajan, 2011, “Super Graceful Labeling for Some Special Graphs,” vol. 9, no. December, pp. 382–404.

N. Hartsfield and G. Ringel, “PEARLS in GRAPH THEORY A Comprehensive Introduction Revised and Augmented.”

G. C. Lau, W. C. Shiu, and H. K. Ng, 2016, “Further results on super graceful labeling of graphs,” *AKCE Int. J. Graphs Comb.*, vol. 13, no. 2, pp. 200–209.