EXISTENCE OF CLEAN ELEMENTS IN A MATRIX RING OVER $\mathbb{Z}$

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Abstrak

A ring R with unity is called clean, if every element $x \in R$ is clean i.e. for every element $x \in R$ there exist an idempotent element $e \in R$ and a unit element $u \in R$ such that $x = e + u$. The set of matrices $X_3(R) = \left\{ \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix} \middle| a_{ij} \in R \right\}$ is a ring with respect to the usual addition and multiplication operation of matrices. The ring $X_3(R)$ is not clean for some integral domain R, but the ring has eight forms of clean elements. Those clean elements are constructed by adding an idempotent element and a unit element. Since the ring $\mathbb{Z}$ of all integers is an integral domain, so the subring $X_3(\mathbb{Z}) \subseteq X_3(R)$ is not clean. In his paper, we discuss the existence of clean elements in $X_3(\mathbb{Z})$ based on the forms of clean elements in $X_3(R)$ and proved the sufficiency and necessary condition of clean elements in $X_3(\mathbb{Z})$.

Key Word: *Clean Elements; Matrix Ring;*

1. INTRODUCTION

In this paper, the ring of matrices 3×3 over a ring R is denoted by $M_3(R) = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \middle| a_{ij} \in R \right\}$. For a ring R, the set of all the unit elements, idempotent elements, and nilpotent elements in R are denoted by $U(R), id(R)$ and $Nil(R)$, respectively. So, $U(R) = \{a \in R | ab = e = ba, \text{for some } b \in R\}$, $id(R) = \{a \in R | a^2 = a\}$, and $Nil(R) = \{a \in R | a^x = 0, \text{for some } x \in R\}$. References (Yang X, 2009 : 157-173) says that R is ring with unity 1, an element $a \in R$ is called clean if a can be written by $a = e + u$, for some $e \in id(R)$ and $u \in U(R)$. A ring R is called clean ring if every element in R is clean.

Meanwhile, in (Diesl AJ, 2013 : 197-211) says that an element a of ring R is called nil-clean, if a can be written by $a = e + n$, where $e \in id(R)$ and $n \in Nil(R)$. In (Faikar M, et al) , discuss about a subset of $M_3(R)$ that nil-clean but not clean. Namely, the subset $X_3(R) = \left\{ \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix} \middle| a_{ij} \in R \right\}$. That means, in subset $X_3(R)$ there exist an element that not clean. The result in (Ambarsari, et al) was showing the eight forms of clean elements in the ring. The clean elements are constructed by adding an idempotent element and a unit element. Since the ring $\mathbb{Z}$ is an integral domain, so the matrix ring $X_3(\mathbb{Z}) = \left\{ \begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix} \middle| a_{ij} \in \mathbb{Z} \right\}$ is not clean.

In this paper, we continue to discuss the existence of clean elements in $X_3(\mathbb{Z})$ based on the forms of

the clean elements in $X_3(R)$ and construct a lemma to proved a theorem. In Section 2, we will use the eight forms in (Ambarsari, et al) to constructed a set of the clean elements of matrix ring $X_3(\mathbb{Z})$ and proved the sufficiency and necessary condition of clean elements in $X_3(\mathbb{Z})$.

As usual, $\mathbb{Z}$ denotes the ring of integers.

2. RESULTS OF THE STUDY

In this section, we construct a set of the clean elements of matrix ring $X_3(\mathbb{Z})$. The set will be used to proved the sufficiency and necessary condition of clean elements in $X_3(\mathbb{Z})$.

This two following lemma explain about the idempotent element and unit element in $X_3(\mathbb{Z})$.

Lemma 2.1 A set of all the idempotent elements in $X_3(\mathbb{Z})$ is

$$id(X_3(\mathbb{Z})) = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & x \end{bmatrix}, \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & 1-a \end{bmatrix} \middle| x, c \in id(\mathbb{Z}), bd = a - a^2 \right\}$$

Proof: See in (Ambarsari, et al) .

Lemma 2.2 A set of all the unit elements in $X_3(\mathbb{Z})$ is $U\big(X_3(\mathbb{Z})\big) = B_1 \cup B_2$, where

$$B_1 = \left\{ \begin{bmatrix} y & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & y \end{bmatrix}, \begin{bmatrix} p & 0 & q \\ 0 & r & 0 \\ s & 0 & -p \end{bmatrix} \middle| y, r \in U(\mathbb{Z}), qs = 1 - p^2 \right\}$$

and

$$B_2 = \left\{ \begin{bmatrix} t & 0 & -q \\ 0 & r & 0 \\ -s & 0 & p \end{bmatrix}, \begin{bmatrix} -t_1 & 0 & q_1 \\ 0 & r & 0 \\ s_1 & 0 & -p_1 \end{bmatrix} \middle| r \in U(\mathbb{Z}), (pt - qs) = 1, (p_1 t_1 - q_1 s_1) = -1 \right\}$$

Proof: See in (Ambarsari, et al) .

Lemma 2.3 The matrix A has invers if and only if $\det(A) \in U(R)$.

Proof: See in (Chapman, 1992) .

Furthermore, we will proved the sufficiency and necessary condition of clean elements in $X_3(\mathbb{Z})$.

Theorem 2.4 The matrix $C = \begin{bmatrix} m & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & m \end{bmatrix} \in X_3(\mathbb{Z})$ is clean if and only if $m, n \in \{-1,0,1,2\}$.

Proof:

($\Longrightarrow$) Let the matrix $C = \begin{bmatrix} m & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & m \end{bmatrix} \in X_3(\mathbb{Z})$ be a clean. We will show that $m, n \in \{-1,0,1,2\}$.

Since C is clean, so $C = E + V$, for some $E \in id(X_3(\mathbb{Z}))$ and $V \in U(X_3(\mathbb{Z}))$.

Based on Lemma 1.1, we get

$$E = \begin{bmatrix} x & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & x \end{bmatrix} \text{ or } E = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & 1-a \end{bmatrix}$$

For some $x, c \in id(\mathbb{Z})$ and $bd = a - a^2$.

We assume that $E = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & 1-a \end{bmatrix}$, where $c \in id(\mathbb{Z}), bd = a - a^2$.

$$V = C - E = \begin{bmatrix} m & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & m \end{bmatrix} - \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & 1-a \end{bmatrix}$$

$$= \begin{bmatrix} m-a & 0 & -b \\ 0 & n-c & 0 \\ -d & 0 & m+a-1 \end{bmatrix}$$

and

$\det(V) = (n-c)\big((m-a)(m+a-1) - bd\big) = (n-c)(m^2 - m)$

Since $V \in U(X_3(\mathbb{Z}))$, then $\det(V) = 1$ or $\det(V) = -1$.

Suppose that $\det(V) = (n-c)(m^2 - m) = 1$. So we have 2 cases:

Case 1: $n - c = 1$ and $m^2 - m = m(m-1) = 1$

Case 1.1: $m = 1$ and $m - 1 = 1$. So, $m = 1$ and $m = 2$. There is a contradiction.

Case 1.2: $m = -1$ and $m - 1 = -1$. So, $m = -1$ and $m = 0$. There is a contradiction.

Case 2: $n - c = -1$ and $m^2 - m = m(m-1) = -1$

Case 2.1: $m = 1$ and $m - 1 = -1$. So, $m = 1$ and $m = 0$. There is a contradiction.

Case 2.2: $m = -1$ and $m - 1 = 1$. So, $m = -1$ and $m = 2$. There is a contradiction.

Therefore, $\det(V) = (n-c)(m^2 - m) = -1$, and we have 2 cases again:

Case 1: $n - c = 1$ and $m^2 - m = m(m-1) = -1$

Case 1.1: $m = 1$ and $m - 1 = -1$. So, $m = 1$ and $m = 0$. There is a contradiction.

Case 1.2: $m = -1$ and $m - 1 = 1$. So, $m = -1$ and $m = 2$. There is a contradiction.

Case 2: $n-c=-1$ and $m^2-m=m(m-1)=1$

Case 2.1: $m=1$ and $m-1=1$. So, $m=1$ and $m=2$. There is a contradiction.

Case 2.2: $m=-1$ and $m-1=-1$. So, $m=-1$ and $m=0$. There is a contradiction.

Therefore, the assuming the form of the matrix E as above is wrong. So, we have the following form of the matrix

$$E=\begin{bmatrix}x & 0 & 0\\ 0 & c & 0\\ 0 & 0 & x\end{bmatrix} \text{ where } x,c\in id(\mathbb{Z})$$

Therefore, we have the unit matrix W as follows

$$W=C-E=\begin{bmatrix}m & 0 & 0\\ 0 & n & 0\\ 0 & 0 & m\end{bmatrix}-\begin{bmatrix}x & 0 & 0\\ 0 & c & 0\\ 0 & 0 & x\end{bmatrix}=\begin{bmatrix}m-x & 0 & 0\\ 0 & n-c & 0\\ 0 & 0 & m-x\end{bmatrix}$$

Since $W\in U(X_3(\mathbb{Z}))$, we have $\det(W)=1$ or $\det(W)=-1$, so we have

$$(m-x)^2(n-c)=1 \text{ or } (m-x)^2(n-c)=-1$$

Therefore, since $m,n,x,c\in\mathbb{Z}$, so $(m-x)^2=\pm1$ and $(n-c)=1$ or $(n-c)=-1$.

Furthermore, we get $m=x\pm1$ and $n=c\pm1$. Since $x,c\in id(\mathbb{Z})$, so $x,c\in\{0,1\}$ and we have $m,n\in\{-1,0,1,2\}$.

($\Longleftarrow$) It is easy to proving this direction.

Theorem 2.5 The matrix $C=\begin{bmatrix}k & 0 & k+1\\ 0 & m & 0\\ -k & 0 & -(k+1)\end{bmatrix}\in X_3(\mathbb{Z})$ is clean if and only if $k\in\mathbb{Z}$,
$m\in\{-1,0,1,2\}$.

Proof :

($\Longrightarrow$) Let the matrix $C=\begin{bmatrix}k & 0 & k+1\\ 0 & m & 0\\ -k & 0 & -(k+1)\end{bmatrix}$ be a clean. We will show that $k\in\mathbb{Z},m\in\{-1,0,1,2\}$.

Since C is clean, so $C=E+V$, for some $E\in id(X_3(\mathbb{Z}))$ and $V\in U(X_3(\mathbb{Z}))$.

Based on Lemma 1.1, we get

$$E=\begin{bmatrix}x & 0 & 0\\ 0 & c & 0\\ 0 & 0 & x\end{bmatrix} \text{ or } E=\begin{bmatrix}a & 0 & b\\ 0 & c & 0\\ d & 0 & 1-a\end{bmatrix}$$

For some $x,c\in id(\mathbb{Z})$ and $bd=a-a^2$.

We assume that $E=\begin{bmatrix}x & 0 & 0\\ 0 & c & 0\\ 0 & 0 & x\end{bmatrix}$, where $x,c\in id(\mathbb{Z})$. So we get

$$V = C - E = \begin{bmatrix} k & 0 & k+1 \\ 0 & m & 0 \\ -k & 0 & -(k+1) \end{bmatrix} - \begin{bmatrix} x & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & x \end{bmatrix}$$
$$= \begin{bmatrix} k-x & 0 & k+1 \\ 0 & m-c & 0 \\ -k & 0 & -(k+x+1) \end{bmatrix}$$

and

$$\det(V) = (m-c)\big((k-x)(-k-x-1) + k(k+1)\big) = (m-c)(x^2+x)$$
$$= \pm 1$$

Since $x, c \in id(\mathbb{Z})$, we have $x^2 + x = \pm 1$, but this is a contradiction for all $x \in \mathbb{Z}$. That means the form of the idempotent matrix E is

$$E = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & 1-a \end{bmatrix}, \text{ where } c \in id(\mathbb{Z}), bd = a - a^2$$

Therefore, we have the unit matrix V as follows

$$V = C - E = \begin{bmatrix} k & 0 & k+1 \\ 0 & m & 0 \\ -k & 0 & -(k+1) \end{bmatrix} - \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & 1-a \end{bmatrix}$$
$$= \begin{bmatrix} k-a & 0 & k-b+1 \\ 0 & m-c & 0 \\ -k-d & 0 & -k+a-2 \end{bmatrix}$$

If $V \in U(X_3(\mathbb{Z}))$, then $\det(V) \in U(\mathbb{Z})$.

$$\det(V) = (m-c)((k-a)(-k+a-2) - (-k-d)(k-b+1))$$
$$= (m-c)(2ak - k + a - bk + dk + d)$$

Case 1: If $\det(V) = 1$, then $(m-c)(2ak - k + a - bk + dk + d) = 1$

Case 1.1: $m - c = 1$ and $2ak - k + a - bk + dk + d = 1$

$$2\text{ak} - k + a - bk + dk + d = 1$$
$$(k+1)(a+d-1) = -k(a-b)$$

So we have $k = a - b - 1$ and $k = -a - d + 1$.

Since $a, b, d \in \mathbb{Z}$, so $k \in \mathbb{Z}$.

Case 1.2: $m - c = -1$ and $2ak - k + a - bk + dk + d = -1$

$$2ak - k + a - bk + dk + d = -1$$
$$k(2a + d - b) - b + a + (a + d) = k - b + a - 1$$

So we have $k \in \mathbb{Z}$.

Case 2: If $\det(V) = -1$, then $(m-c)(2ak - k + a - bk + dk + d) = -1$

Case 1.1: $m - c = 1$ and $2ak - k + a - bk + dk + d = -1$

$$2ak - k + a - bk + dk + d = -1$$
$$k(2a + d - b) - b + a + (a + d) = k - b + a - 1$$

So we have $k \in \mathbb{Z}$.

Case 1.2: $m - c = -1$ and $2ak - k + a - bk + dk + d = 1$

$$2ak - k + a - bk + dk + d = 1$$
$$(k+1)(a+d-1) = -k(a-b)$$

So we have $k = a - b - 1$ and $k = -a - d + 1$.

Since $a, b, d \in \mathbb{Z}$, so $k \in \mathbb{Z}$.

Since $c \in id(\mathbb{Z})$, so we get $m \in \{-1,0,1,2\}$.

($\Longleftarrow$) It is easy to proving this direction.

Theorem 2.6 The matrix $C = \begin{bmatrix} 1+k & 0 & 1+k \\ 0 & m & 0 \\ 1-k & 0 & 1-k \end{bmatrix} \in X_3(\mathbb{Z})$ is clean if and only if $k \in \mathbb{Z}$,
$m \in \{-1,0,1,2\}$.

Proof :

($\Longrightarrow$) Let the matrix $C = \begin{bmatrix} 1+k & 0 & 1+k \\ 0 & m & 0 \\ 1-k & 0 & 1-k \end{bmatrix}$ be a clean. We will show that $k \in \mathbb{Z}, m \in \{-1,0,1,2\}$.

Since C is clean element, then $C = E + U$, for some $E \in id(X_3(\mathbb{Z}))$ and $V \in U(X_3(\mathbb{Z}))$.

Based on Lemma 1.1, we get

$$E = \begin{bmatrix} x & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & x \end{bmatrix} \text{ or } E = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & 1-a \end{bmatrix}$$

For some $x, c \in id(\mathbb{Z})$ and $bd = a - a^2$.

We assume that $E = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & 1-a \end{bmatrix}$ where $c \in id(\mathbb{Z}), bd = a - a^2$.

So we get

$$V = C - E = \begin{bmatrix} 1+k & 0 & 1+k \\ 0 & m & 0 \\ 1-k & 0 & 1-k \end{bmatrix} - \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & 1-a \end{bmatrix}$$

$$= \begin{bmatrix} 1+k-a & 0 & 1+k-b \\ 0 & m-c & 0 \\ 1-k-d & 0 & -k+a \end{bmatrix}$$

If $V \in U(X_3(\mathbb{Z}))$, then $\det(V) = 1$ or $\det(V) = -1$, so we have $\det(V) = (m-c)(2ak - bk + dk - k + b + d - 1)$

Suppose that $\det(V) = (m-c)(2ak - bk + dk - k + b + d - 1) = 1$. So we have 2 cases:

Case 1: $m - c = 1$ and $2ak - bk + dk - k + b + d - 1 = 1$

$$2ak - bk + dk - k + b + d - 2 + 1 = 1$$

$$k(2a - b + d - 1) + (b + d - 2) + 1 = k(0) + (0) + 1$$

then $2a - b + d - 1 = 0$ and $b + d - 2 = 0$

and we have $b = 2 - d$, next substitution

$2a - b + d - 1 = 0$

$2(a + d) = 3$

Since $\mathbb{Z}$ is integral domain, so $\forall a, d \in \mathbb{Z}$ such that $2(a + d) \neq 3$.

Case 2: $m - c = -1$ and $2ak - bk + dk - k + b + d - 1 = -1$

$2ak - bk + dk - k + b + d - 1 = -1$

$k(2a-1)-b(k-1)+d(k+1)-1$
$= k(0)-b(0)+d(0)-1$
then $2a-1=0, k-1=0$ and $k+1=0$
so we have $k=1$ and $k=-1$. There is a contradiction.

Therefore, $\det(V)=(m-c)(2ak-bk+dk-k+b+d-1)=-1$, we have 2 cases again:

Case 1: $m-c=1$ and $2ak-bk+dk-k+b+d-1=-1$
$2ak-bk+dk-k+b+d-1=-1$
$k(2a-1)-b(k-1)+d(k+1)-1$
$= k(0)-b(0)+d(0)-1$
then $2a-1=0, k-1=0$ and $k+1=0$
so we have $k=1$ and $k=-1$. There is a contradiction.

Case 2: $m-c=-1$ and $2ak-bk+dk-k+b+d-1=1$
$2ak-bk+dk-k+b+d-1=1$
$k(2a-b+d-1)+(b+d-2)+1=k(0)+(0)+1$
then $2a-b+d-1=0$ and $b+d-2=0$
so we have $b=2-d$, next substitution
$2a-b+d-1=0$
$2(a+d)=3$

Since $\mathbb{Z}$ is integral domain, so $\forall \mathrm{a}, \mathrm{d} \in \mathbb{Z}$ such that $2(\mathrm{a}+\mathrm{d}) \neq 3$.

Therefore, the assuming the form of the matrix Eas above is wrong. So, we have the following form of the matrix

$$E=\begin{bmatrix} x & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & x \end{bmatrix} \text{ where } x, c \in id(\mathbb{Z})$$

Therefore, we have the unit matrix W as follows

$$W=C-E=\begin{bmatrix} 1+k & 0 & 1+k \\ 0 & m & 0 \\ 1-k & 0 & 1-k \end{bmatrix}-\begin{bmatrix} x & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & x \end{bmatrix}$$
$$=\begin{bmatrix} 1+k-x & 0 & 1+k \\ 0 & m-c & 0 \\ 1-k & 0 & 1-k-x \end{bmatrix}$$

If $\det(W) \in U(X_3(\mathbb{Z}))$, then $\det(W)=1$ or $\det(W)=-1$, so we have
$\det(W)=(m-c)\big((1+k-x)(1-k-x)-(1-k)(1+k)\big)$
$=(m-c)(x^2-2x)$

Since $x, c \in id(\mathbb{Z})$, so we get

Case 1: for $x=0$ such that $x^2-2x-1 \neq 0$ and $x^2-2x+1 \neq 0$,

Case 2: for $x=1$ such that $x^2-2x+1=0$ but $x^2-2x-1 \neq 0$.

So, $x=1$ and we have

$$W=\begin{bmatrix} 1+k-1 & 0 & 1+k \\ 0 & m-c & 0 \\ 1-k & 0 & 1-k-1 \end{bmatrix}=\begin{bmatrix} k & 0 & 1+k \\ 0 & m-c & 0 \\ 1-k & 0 & -k \end{bmatrix}$$

Since $c \in id(\mathbb{Z})$, so $m \in \{-1,0,1,2\}$. So, we get $k \in \mathbb{Z}$.

($\Longleftarrow$) It is easy to proving this direction.

Theorem 2.7 The matrix $C = \begin{bmatrix} k & 0 & k+1 \\ 0 & m & 0 \\ 2-k & 0 & 1-k \end{bmatrix} \in X_3(\mathbb{Z})$ is clean if and only if $k = -1$,
$m \in \{-1,0,1,2\}$.

Proof :

$(\Longrightarrow)$ Let the matrix $C = \begin{bmatrix} k & 0 & k+1 \\ 0 & m & 0 \\ 2-k & 0 & 1-k \end{bmatrix}$ be a clean. We will show that $k = -1, m \in \{-1,0,1,2\}$.

Since C is clean, then $C = E + V$, for some $E \in id(X_3(\mathbb{Z}))$ and $V \in U(X_3(\mathbb{Z}))$.

Based on Lemma 1.1, we get

$$E = \begin{bmatrix} x & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & x \end{bmatrix} \text{ or } E = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & 1-a \end{bmatrix}$$

For some $x, c \in id(\mathbb{Z})$ and $bd = a - a^2$.

We assume that $E = \begin{bmatrix} x & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & x \end{bmatrix}$ where $x, c \in id(\mathbb{Z})$. So we get

$$V = C - E = \begin{bmatrix} k & 0 & k+1 \\ 0 & m & 0 \\ 2-k & 0 & 1-k \end{bmatrix} - \begin{bmatrix} x & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & x \end{bmatrix}$$

$$= \begin{bmatrix} k-x & 0 & 1+k \\ 0 & m-c & 0 \\ 2-k & 0 & 1-k-x \end{bmatrix}$$

$$\det(V) = (m-c)\big((k-x)(1-k-x) - (2-k)(1+k)\big)$$
$$= (m-c)(x^2 - x - 2)$$

Since $V \in U(X_3(\mathbb{Z}))$, then $\det(V) = 1$ or $\det(V) = -1$.

Since $x, c \in id(\mathbb{Z})$, so we have $\forall x \in \mathbb{Z}$ such that $x^2 + x - 3 \neq 0$ and $x^2 + x - 1 \neq 0$.

Therefore, the assuming the form of the matrix E as above is wrong. So, we have the following form of the matrix

$$E = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & 1-a \end{bmatrix} \text{ where } c \in id(\mathbb{Z}), bd = a - a^2.$$

Therefore, we have the unit matrix W as follows

$$W = C - E = \begin{bmatrix} k & 0 & k+1 \\ 0 & m & 0 \\ 2-k & 0 & 1-k \end{bmatrix} - \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & 1-a \end{bmatrix}$$

$$= \begin{bmatrix} k-a & 0 & k+1-b \\ 0 & m-c & 0 \\ 2-k-d & 0 & -k+a \end{bmatrix}$$

If $W \in U(X_3(\mathbb{Z}))$, then $\det(W) \in U(\mathbb{Z})$.

$$\det(W) = (m-c)((k-a)(-k+a) - (2-k-d)(k+1-b))$$
$$= (m-c)(2ak - k - 2 + 2b - bk + dk + d - a)$$

Case 1: If $\det(W) = 1$, then $(m-c)(2ak - k - 2 + 2b - bk + dk + d -$

$a) = 1$
Case 1.1: $m - c = 1$ and $2ak - k - 2 + 2b - bk + dk + d - a = 1$

$2ak - k - 2 + 2b - bk + dk + d - a = 1$

$k(2a - 1 - b + d) + (-3 + 2b + d - a) + 1 = k(0) + (0) + 1$

So we get $3(b - a) = 2$, so $\forall a, b \in \mathbb{Z}$ such that $3(b - a) \neq 2$.

Case 1.2: $m - c = -1$ and $2ak - k - 2 + 2b - bk + dk + d - a = -1$

$2ak - k - 2 + 2b - bk + dk + d - a = -1$

$k(2a - b + d - 1) + (-a + 2b + d - 1) + 3a - 3b = 3a - 3b$

So we get $k = -1$.

Case 2: If $\det(W) = -1$, then $(m - c)(2ak - k - 2 + 2b - bk + dk + d - a) = -1$

Case 2.1: $m - c = 1$ and $2ak - k - 2 + 2b - bk + dk + d - a = -1$

$2ak - k - 2 + 2b - bk + dk + d - a = -1$

$k(2a - b + d - 1) + (-a + 2b + d - 1) + 3a - 3b = 3a - 3b$

So we get $k = -1$.

Case 2.2: $m - c = -1$ and $2ak - k - 2 + 2b - bk + dk + d - a = 1$

$2ak - k - 2 + 2b - bk + dk + d - a = 1$

$k(2a - b + d) + (a - 2b + d - 3) + 3a - 3b = 3a - 3b$

So we get $3(b - a) = 2$, so $\forall a, b \in \mathbb{Z}$ such that $3(b - a) \neq 2$.

Since $c \in id(\mathbb{Z})$, so we get $m \in \{-1,0,1,2\}$.

($\Longleftarrow$) It is easy to proving this direction.

Theorem 2.8 The matrix $C = \begin{bmatrix} 1+k & 0 & k \\ 0 & m & 0 \\ n-1 & 0 & n+1 \end{bmatrix} \in X_3(\mathbb{Z})$ is clean if and only if $(1) k = 1, n \in \mathbb{Z}$ and $m \in \{-1,0,1,2\}$, $(2) k = -1, n \in \mathbb{Z}$ and $m \in \{-1,0,1,2\}$, $(3) n = -2k, k \in \mathbb{Z}$ and $m \in \{-1,0,1,2\}$, or $(4) n = -2k - 2, k \in \mathbb{Z}$ and $m \in \{-1,0,1,2\}$.

Proof:

($\Longrightarrow$) Let the matrix $C = \begin{bmatrix} 1+k & 0 & k \\ 0 & m & 0 \\ n-1 & 0 & n+1 \end{bmatrix} \in X_3(\mathbb{Z})$ be a clean.

We will show that $(1) k = 1, n \in \mathbb{Z}$ and $m \in \{-1,0,1,2\}$, $(2) k = -1, n \in \mathbb{Z}$ and
$m \in \{-1,0,1,2\}$, $(3) n = -2k, k \in \mathbb{Z}$ and $m \in \{-1,0,1,2\}$, or $(4) n = -2k - 2, k \in \mathbb{Z}$ and
$m \in \{-1,0,1,2\}$

Since C is clean, then $C = E + V$, for some $E \in id(X_3(\mathbb{Z}))$ and $V \in U(X_3(\mathbb{Z}))$.

Based on Lemma 1.1, we get

$$E = \begin{bmatrix} x & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & x \end{bmatrix} \text{ or } E = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & 1-a \end{bmatrix}$$

For some $x, c \in id(\mathbb{Z})$ and $bd = a - a^2$.

We assume that $E=\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & 1-a \end{bmatrix}$ where $c \in id(\mathbb{Z}), bd = a - a^2$.

So we get

$$V = C - E = \begin{bmatrix} 1+k & 0 & k \\ 0 & m & 0 \\ n-1 & 0 & n+1 \end{bmatrix} - \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & 1-a \end{bmatrix}$$
$$= \begin{bmatrix} 1+k-a & 0 & k-b \\ 0 & m-c & 0 \\ n-1-d & 0 & n+a \end{bmatrix}$$

If $V \in U(X_3(\mathbb{Z}))$, then $\det(V) = 1$ or $\det(V) = -1$, so we have

Suppose that $\det(V) = (m-c)(-na + bn + n + ak + dk + k - b) = 1$.

Case 1: $m - c = 1$ and $-na + bn + n + ak + dk + k - b = 1$

$n(-a+b+1) + k(a+d+1) + (-b-1) = n(0) + k(0) + 0$

So we have $a = 0, b = -1$ and $d = -1$

Case 2: $m - c = -1$ and $-na + bn + n + ak + dk + k - b = -1$

$n(-a+b+1) + k(a+d+1) + (-b+1) = n(0) + k(0) + 0$

So we have $a = 2, b = 1$ and $d = -3$

Therefore, $\det(V) = (m-c)(-na + bn + n + ak + dk + k - b) = -1$.

Case 1: $m - c = 1$ and $-na + bn + n + ak + dk + k - b = -1$

$n(-a+b+1) + k(a+d+1) + (-b+1) = n(0) + k(0) + 0$

So we have $a = 2, b = 1$ and $d = -3$

Then we have matrix V:

$$V = \begin{bmatrix} 1+k-a & 0 & k-b \\ 0 & m-c & 0 \\ n-d-1 & 0 & n+a \end{bmatrix} = \begin{bmatrix} k-1 & 0 & k-1 \\ 0 & m-c & 0 \\ n+2 & 0 & n+2 \end{bmatrix}$$

but $\det(V) = (m-c)((k-1)(n+2) - (k-1)(n+2)) = 0 \neq 1$.

There is a contradiction.

Case 2: $m - c = -1$ and $-na + bn + n + ak + dk + k - b = 1$

$n(-a+b+1) + k(a+d+1) + (-b-1) = n(0) + k(0) + 0$

then we have $a = 0, b = -1$ and $d = -1$

So we have matrix V:

$$V = \begin{bmatrix} 1+k-a & 0 & k-b \\ 0 & m-c & 0 \\ n-d-1 & 0 & n+a \end{bmatrix} = \begin{bmatrix} k+1 & 0 & k+1 \\ 0 & m-c & 0 \\ n & 0 & n \end{bmatrix}$$

but $\det(V) = (m-c)\big(n(k+1) - n(k+1)\big) = 0 \neq -1$. There is a contradiction.

Therefore, the assuming the form of the matrix E as above is wrong. So, we have the following form of the matrix

$$E = \begin{bmatrix} x & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & x \end{bmatrix} \text{ where } x, c \in id(\mathbb{Z}).$$

Therefore, we have the unit matrix W as follows

$$W = C - E = \begin{bmatrix} 1+k & 0 & k \\ 0 & m & 0 \\ n-1 & 0 & n+1 \end{bmatrix} - \begin{bmatrix} x & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & x \end{bmatrix}$$
$$= \begin{bmatrix} 1+k-x & 0 & k \\ 0 & m-c & 0 \\ n-1 & 0 & n+1-x \end{bmatrix}$$

If $W \in U(X_3(\mathbb{Z}))$, then $\det(W) = 1$ or $\det(W) = -1$, then

Case 1: Let $\det(W) = -1$, then $(m-c)\big((1+k-x)(n-x+1) - k(n-1)\big) = -1$

Case 1.1: $m - c = 1$ and $\big((1+k-x)(n-x+1) - k(n-1)\big) = -1$,

Since $c \in id(\mathbb{Z})$, so we get $m \in \{1,2\}$.

Since $x \in id(\mathbb{Z})$, so we get $n = -2 - 2k$ or $k = -1$

Case 1.2: $m - c = -1$ and $\big((1+k-x)(n-x+1) - k(n-1)\big) = 1$

Since $c \in id(\mathbb{Z})$, so we get $m \in \{-1,0\}$.

Since $x \in id(\mathbb{Z})$, so we get $n = -2k$ or $k = 1$

Case 2: Let $\det(W) = 1$, then $(m-c)\big((1+k-x)(n-x+1) - k(n-1)\big) = 1$

Case 2.1: $m - c = 1$ and $\big((1+k-x)(n-x+1) - k(n-1)\big) = 1$,

Since $c \in id(\mathbb{Z})$, so we get $m \in \{1,2\}$.

Since $x \in id(\mathbb{Z})$, so we get $n = -2k$ or $k = 1$

Case 2.2: $m - c = 1$ and $\big((1+k-x)(n-x+1) - k(n-1)\big) = -1$,

Since $c \in id(\mathbb{Z})$, so we get $m \in \{-1,0\}$.

Since $x \in id(\mathbb{Z})$, so we get $n = -2 - 2k$ or $k = -1$

($\Leftarrow$) It is easy to proving this direction.

Theorem 2.9 The matrix $C = \begin{bmatrix} 1+k & 0 & 1+n \\ 0 & m & 0 \\ -k & 0 & 1-n \end{bmatrix} \in X_3(\mathbb{Z})$ is clean if and only if

(1) $k \in \{-2,-1,1,2\}, n \in \mathbb{Z}$ and $m \in \{-1,0,1,2\}$, (2) $n = 2k, k \in \mathbb{Z}$ and $m \in \{-1,0,1,2\}$, or

(3) $n = 2k+2, k \in \mathbb{Z}$ and $m \in \{-1,0,1,2\}$.

Proof:

($\Rightarrow$) Let the matrix $C = \begin{bmatrix} 1+k & 0 & 1+n \\ 0 & m & 0 \\ -k & 0 & 1-n \end{bmatrix} \in X_3(\mathbb{Z})$ be a clean.

We will show that (1) $k \in \{-2,-1,1,2\}, n \in \mathbb{Z}$ and $m \in \{-1,0,1,2\}$, (2) $n = 2k, k \in \mathbb{Z}$ and
$m \in \{-1,0,1,2\}$, or (3) $n = 2k+2, k \in \mathbb{Z}$ and $m \in \{-1,0,1,2\}$.

Since C is clean, so $C = E + V$, for some $E \in id(X_3(\mathbb{Z}))$ and $V \in U(X_3(\mathbb{Z}))$.

Based on Lemma 1.1, we get

$$E = \begin{bmatrix} x & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & x \end{bmatrix} \text{ or } E = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & 1-a \end{bmatrix}$$

For some $x, c \in id(\mathbb{Z})$ and $bd = a - a^2$.

Case 1: Suppose that $E = \begin{bmatrix} x & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & x \end{bmatrix}$, where $x, c \in id(\mathbb{Z})$. So we get

$$V = \begin{bmatrix} 1+k & 0 & 1+n \\ 0 & m & 0 \\ -k & 0 & 1-n \end{bmatrix} - \begin{bmatrix} x & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & x \end{bmatrix} = \begin{bmatrix} 1+k-x & 0 & 1+n \\ 0 & m-c & 0 \\ -k & 0 & 1-n-x \end{bmatrix}$$

If $V \in U(X_3(\mathbb{Z}))$, then $\det(V) = 1$ or $\det(V) = -1$, so we have

$\det(V) = (m-c)(1-n-2x+2k-xk+nx+x^2)$

Suppose that $\det(V) = (m-c)(1-n-2x+2k-xk+nx+x^2) = 1$.

Case 1.1: $m - c = 1$ and $1-n-2x+2k-xk+nx+x^2 = 1$.

Since $x \in id(\mathbb{Z})$, so $n = 2k$ or $k = 1$.

Case 1.2: $m - c = -1$ and $1-n-2x+2k-xk+nx+x^2 = -1$

Since $x \in id(\mathbb{Z})$, so $n = 2k+2$ or $k = -1$.

Suppose that $\det(V) = (m-c)(1-n-2x+2k-xk+nx+x^2) = -1$.

Case 2.1: $m - c = 1$ and $1-n-2x+2k-xk+nx+x^2 = -1$.

Since $x \in id(\mathbb{Z})$, so $n = 2k+2$ or $k = -1$.

Case 2.2: $m - c = -1$ and $1-n-2x+2k-xk+nx+x^2 = 1$

Since $x \in id(\mathbb{Z})$, so $n = 2k$ or $k = 1$.

Since $c \in id(\mathbb{Z})$, so we get $m \in \{-1,0,1,2\}$.

Case 2: Suppose that $E = \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & 1-a \end{bmatrix}$, where $c \in id(\mathbb{Z}), bd = a - a^2$. So we get

$$V = \begin{bmatrix} 1+k & 0 & 1+n \\ 0 & m & 0 \\ -k & 0 & 1-n \end{bmatrix} - \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & 1-a \end{bmatrix}$$

$$= \begin{bmatrix} 1+k-a & 0 & 1+n-b \\ 0 & m-c & 0 \\ -k-d & 0 & -n+a \end{bmatrix}$$

If $V \in U(X_3(\mathbb{Z}))$, then $\det(V) = 1$ or $\det(V) = -1$, so we have

$\det(V) = (m-c)(an+ak-bk+k-n+nd+d)$

Suppose that $\det(V) = (m-c)(an+ak-bk+k-n+nd+d) = 1$.

Case 1.1: $m - c = 1$ and $an+ak-bk+k-n+nd+d = 1$

$an+ak-bk+k-n+nd+d = 1$

$an+dn-kb-n = -ak-d-k+1$

So we get $k = 1$ and $n = -1$.

Case 1.2: $m - c = -1$ and $an+ak-bk+k-n+nd+d = -1$

$an+ak-bk+k-n+nd+d = 1$

$n(a+1)+(ak+k-bk+2) = n(2-d)$

So we get $k = 2$ or $k = -2$.

Suppose that $\det(V) = (m-c)(an+ak-bk+k-n+nd+d) = -1$.

Case 2.1: $m - c = 1$ and $an+ak-bk+k-n+nd+d = -1$.

$an+ak-bk+k-n+nd+d = 1$

$n(a+1)+(ak+k-bk+2)=n(2-d)$

So we get $k=2$ or $k=-2$.

Case 2.2: $m-c=-1$ and $an+ak-bk+k-n+nd+d=1$

$an+ak-bk+k-n+nd+d=1$

$an+dn-kb-n=-ak-d-k+1$

So we get $k=1$ and $n=-1$.

Since $c\in id(\mathbb{Z})$, so we get $m\in\{-1,0,1,2\}$.

($\Longleftarrow$) It is easy to proving this direction

3. References

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