SOME PROPERTIES OF *M*-ACCRETIVE OPERATORS IN NORMED SPACES

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Abstract

Multivalued mapping in normed space has been extensively studied in Mathematical analysis. One example of multivalued mapping is normalized duality mapping. This mapping leads us to another example of multivalued operator, named accretive operator, which is also a multivalued operator. This study was aimed to examine the basic concepts of accretive and m-accretive operator. Furthermore, discussion on some properties of accretive and m-accretive operator was provided.

Keywords: normalized duality mapping, accretive operator, m-accretive operator.

Presenting Author's Biography

Christina Kartika Sari. She studied in mathematics education and was graduated from Universitas Sebelas Maret on 2011 for her bachelor. However, for her post-graduate program, she focused on pure mathematics, especially mathematical analysis. She has been teaching in Universitas Muhammadiyah Surakarta after she finished her post-graduate program from Universitas Gadjah Mada on 2015.

1. Introduction

The definition of a multivalued mapping would be provided at first. Given $2^{\mathbb{R}} = \{X: X \subseteq \mathbb{R}\}$ and a mapping $A: \mathbb{R} \to 2^{\mathbb{R}}$, where $A(x) = [0, |x|]$ for every $x \in \mathbb{R}$. We know that for every $x \in \mathbb{R}$, $A(x)$ is a closed interval. Hence, range of A is a set of closed intervals. It is one example of multivalued mappings. Let X and Y be nonempty sets. A mapping that maps X to 2^Y is called multivalued mapping. If A is a multivalued mapping, then domain of A is $D(A)$ and range of A is $\mathcal{R}(A)$, where:

$$\mathcal{R}(A) = \bigcup_{x \in D(A)} A(x).$$

Furthermore, if both of X and Y are vector spaces, then A is called multivalued operator.

Let X be a real normed space. If a multivalued operator A maps to singleton, we say it as a singlevalued operator. It is clear that an operator $I: X \to 2^X$, where $I(x) = \{x\}$ for any $x \in X$, is a singlevalued operator. In the sequel, we say that I is an identity operator. This paper is going to study other examples of multivalued operators and some of their properties in normed spaces.

This was a literature study of some papers related to an operator in normed space, named m-accretive operator. This operator has a role in applied mathematics. In 1967, Browder [1] studied about non-expansive and accretive operator in Banach spaces. He got some result related to accretive operator. Barbu [2] also studied accretive set in Banach spaces. Let X be a real Banach space. In his book, Barbu talks about accretive as a subset of $X \times X$. Furthermore, Barbu also talks about dissipative and m-dissipative set. A dissipative set is negative of an accretive set. It means if A is an acrretive set then $-A$ is a dissipative set. However, an element of subset of $X \times X$ can be assumed as a pair of an element of a mapping domain with its mapping value. Therefore, it makes sense if we assume accretive as an operator. The main contribution of this paper is to review some properties of m-accretive as an operator, not a set. Some properties of this operator will be obtained by assuming properties of accretive set, which is discussed Barbu, as properties of operator. This paper is going to study basic concepts and some properties of accretive and m-accretive operator through exposure proofs and examples in more detail.

2. Preliminary

We are going to continue to another example of multivalued mapping, named normalized duality mapping. Before we study more about normalized duality mapping, we need to know about dual space. Let X be a real normed space. The set of all bounded linear functionals on X is called dual space of X and is denoted by X^*. Some properties related to dual space, that we need to study normalized duality mapping, have been discussed by Kreyszig [3].

Let X be a real normed space and X^* be a dual space of X. In the following, $\langle f, x\rangle$ denotes value of f in $x \in X$ (or $\langle f, x\rangle = f(x)$), where $f \in X^*$. Theorem 1 will ensure that normalized duality mapping is well-defined.

Theorem 1. *Let X be a real normed space and X^* dual space of X. For any $x \in X$, we have:*

$$\{f \in X^*: \langle x, f\rangle = \|x\|^2 = \|f\|^2\} \neq 0.$$

Proof. If $x = 0$, then we have $f = 0 \in X^*$, $f(x) = 0 = \|x\|^2$, and $\|f\| = 0 = \|x\|$. Therefore, $f = 0 \in \{f \in X^*: \langle x, f\rangle = \|x\|^2 = \|f\|^2\}$. If $x \neq 0$, then $x\|x\| \neq 0$. By Hahn-Banach Theorem in Kreyszig [3], there exist $f \in X^*$ such that $\|f\| = 1$ and $f(x\|x\|) = \big\|x\|x\|\big\|$. A mapping $\hat{f}: X \to \mathbb{R}$ defined by

$$\hat{f}(y) = \|x\| f(y)$$

for any $y \in X$. Furthermore, we have

$$\hat{f}(x) = \|x\| f(x) = f(\|x\| x) = \big\|x\|x\|\big\| = \|x\|^2,$$

so that

$$\hat{f}(x) = \|x\|^2. \tag{1}$$

Now, it remains to prove $\|\hat{f}\| = \|x\|$. For any $y \in X$, where $\|y\| \leq 1$, we have

$$|\hat{f}(y)| = |\|x\|f(y)| = \|x\||f(y)| \leq \|x\|\|f\|\|y\| \leq \|x\|.$$

Thus

$$\|\hat{f}\| = \sup\{|\hat{f}(y)|: y \in X, \|y\| \leq 1\} \leq \|x\|. \tag{2}$$

By Eq. (1), it satisfies

$$\|x\|^2 = |\hat{f}(x)| \leq \|\hat{f}\|\|x\|. \tag{3}$$

From Eq. (2) and Eq. (3), the desired conclusion follows.

Under Theorem 1, it makes sense to define a mapping $J: X \to 2^{X^*}$ by

$$J(x) = \{f \in X^*: \langle x, f \rangle = \|x\|^2, \|x\| = \|f\|\}, \text{ for any } x \in X.$$

This is called the normalized duality mapping. Clearly, J is multivalued mapping. The normalized duality mapping is going to lead us to another example of multivalued mapping, named accretive operator, that are going to discuss in the following section.

3. Accretive Operators

Let X be a real normed space and let $\phi: [0, \infty) \to [0, \infty)$ be a continuous mapping, where $\phi(0) = 0$ and $\phi(r) > 0$ for any $r > 0$. An operator $A: D(A) \subseteq X \to 2^X$ is said to be ϕ-expansive if for every $x, y \in D(A)$ obtain

$$\|u - v\| \geq \phi(\|x - y\|) \text{ for all } u \in A(x), v \in A(y).$$

For example, given a mapping $\phi: [0, \infty) \to [0, \infty)$ defined by

$$\phi(x) = 2x \text{ for all } x \in [0, \infty)$$

and an operator $A: \mathbb{R} \to 2^{\mathbb{R}}$ defined by

$$A(x) = \{4x\} \text{ for all } x \in \mathbb{R}.$$

Clearly, A is a ϕ-expansive operator. Meanwhile, an operator $A: D(A) \subseteq X \to 2^X$ is said to be ϕ-expansive if for every $x, y \in D(A)$ there exists $j \in J(x - y)$ such that

$$\langle u - v, j \rangle \geq \phi(\|x - y\|)\|x - y\| \text{ for all } u \in A(x), v \in A(y),$$

where $J: X \to 2^{X^*}$ is the normalized duality mapping. Furthermore, an operator A is said to be strongly accretive if $\phi(r) = kr$ with $0 < k < 1$.

If ϕ is zero mapping, then ϕ is not appropriate with the latter ϕ. In this case, we say the latter mapping A is accretive.

Definition 2. *Let X be a real normed space. An operator $A: D(A) \subseteq X \to 2^X$ is said to be accretive if for every $x, y \in D(A)$ there exists $j \in J(x - y)$ such that*

$$\langle u - v, j \rangle \geq 0 \; for\; all\; u \in A(x), v \in A(y)$$

where $J: X \to 2^{X^*}$ is the normalized duality mapping.

Example 3. Given an operator $T: X = \mathbb{R} \to 2^X$ defined by

$$T(x) = \{x\} \text{ for all } x \in X.$$

For every $x, y \in X$, we have

$$J(x - y) = \{f \in X^*: \langle x - y, f \rangle = |x - y|^2 = \|f\|^2\}.$$

Define a mapping $f: X \to \mathbb{R}$ with $f(t) = (x - y)t$ for any $t \in X$. Because $f \in X^*$ and $\|f\| = |x - y|$, so $\langle x - y, f \rangle = |x - y|^2 = \|f\|^2$. Therefore, $f \in J(x - y)$. Furthermore, it obtains

$$\langle T(x) - T(y), f \rangle = f(T(x) - T(y)) = (x - y)(T(x) - T(y)) = (x - y)^2 > 0.$$

Hence, T is an accretive operator. ■

It is not easy to find an element of value of the normalized duality mapping that satisfies Definition 2. Corollary 5 is going to offer a characteristic of an accretive operator, so can be used to check an operator is accretive or not.

Theorem 4. *Let X be a real normed space and $J: X \to 2^{X^*}$ be normalized duality mapping. For every $x, y \in X$ we have $\|x\| \leq \|x + \alpha y\|$ for all $\alpha > 0$ if and only if there exists $j \in J(x)$ such that $\langle y, j \rangle \geq 0$.*

Proof. If $x = 0$ then for every $y \in X$obtains $\|x\| = 0 \leq \|0 + \alpha y\|$ for all $\alpha > 0$. There exists $j = 0 \in J(0)$ where $J(0) = \{j \in X^*: \langle 0, j \rangle = \|0\|^2 = \|j\|^2\}$. If $x \neq 0$, for every $y \in X$ there exists $j \in J(x)$ so that $\langle y, j \rangle \geq 0$. For any $\alpha > 0$ we have

$$\begin{aligned} \|x\|^2 &= \langle x, j \rangle \\ &\leq \langle x, j \rangle + \alpha \langle y, j \rangle \\ &\leq \langle x + \alpha y, j \rangle \\ &\leq \|j\| \|x + \alpha y\|. \end{aligned}$$

Because $\|x\| = \|j\|$then $\|x\| \leq \|x + \alpha y\|$. Conversely, for every $y \in X$ we have $\|x\| \leq \|x + \alpha y\|$. By Theorem 1, there exists $j_\alpha \in J(x + \alpha y)$, this means $\langle x + \alpha y, j_\alpha \rangle = \|x + \alpha y\|^2 = \|j_\alpha\|^2$. Define $g_\alpha = \frac{j_\alpha}{\|j_\alpha\|}$ so it obtains $\|g_\alpha\| = 1$. Consequently, $g_\alpha \in \frac{1}{\|j_\alpha\|} J(x + \alpha y)$. Therefore

$$\begin{aligned} \|x\| &\leq \|x + \alpha y\| \\ &= \frac{\|x + \alpha y\|^2}{\|j_\alpha\|} \\ &= \frac{\langle x + \alpha y, j_\alpha \rangle}{\|j_\alpha\|} \\ &= \langle x + \alpha y, g_\alpha \rangle \\ &= \langle x, g_\alpha \rangle + \alpha \langle y, g_\alpha \rangle. \end{aligned}$$

A net $(g_\alpha)_{\alpha \in \mathbb{R}^+}$ is on the closed unit ball. According to Brezis [4], closed unit ball is compact in weak* topology, so net $(g_\alpha)_{\alpha \in \mathbb{R}^+}$ has convergent subnet. Let $(\hat{g}_\alpha)_{\alpha \in \mathbb{R}^+}$ be a convergent subnet of net $(g_\alpha)_{\alpha \in \mathbb{R}^+}$ and g be its limit. Clearly, $g \in X^*$.
Notice that

$$\lim_{\alpha \to 0^+} \|x\| \leq \lim_{\alpha \to 0^+} \|x\| + \lim_{\alpha \to 0^+} \alpha \langle y, \hat{g}_\alpha \rangle$$

so

$$\|x\| \leq \langle x, g \rangle. \tag{4}$$

Furthermore, because $\|\hat{g}_\alpha\| = 1$ then

$$\|x\| \leq \langle x, \hat{g}_\alpha \rangle + \alpha \langle y, \hat{g}_\alpha \rangle \leq \|x\| + \alpha \langle y, \hat{g}_\alpha \rangle.$$

Hence, $\alpha \langle y, \hat{g}_\alpha \rangle \geq 0$. Consequently, we have

$$\langle y, g \rangle \geq 0. \tag{5}$$

By Eq. (4) , it obtains $\|x\| \leq \langle x, g \rangle \leq \|x\| \|g\| \leq \|x\|$ so $\langle x, g \rangle = \|x\|$. Define a mapping $j: X \to \mathbb{R}$ by

$$j(w) = g(w)\|x\| \text{ for all } w \in X.$$

By proving that $j \in J(x)$ and $\langle y, j \rangle \geq 0$, proof of this theorem will be complete. It is clear that $j \in X^*$. Furthermore, for any $w \in X$, $\|w\| \leq 1$, we have

$$|\langle w, j \rangle| \leq \|g\| \|w\| \|x\| \leq \|g\| \|x\| = \|x\|,$$

so $\|j\| = \sup\{|\langle w, j\rangle| : w \in X, \|w\| \leq 1\} \leq \|x\|$. Because $\|x\| \leq \langle x, g\rangle$, so we have

$$\|x\|^2 \leq \langle x, g\rangle \|x\| = \langle x, j\rangle \leq \|x\|\|j\|.$$

Therefore $\|x\| \leq \|j\|$. Consequently, $\|x\| = \|j\|$. This means $j \in J(x)$. Furthermore, by Eq. (5) we have $\langle y, j\rangle = \langle y, g\rangle \|x\| \geq 0$. ■

Based on Theorem 4, we can get Corollary 5 that declare a characteristic of accretive operator.

Corollary 5. *Let X be a real normed space. An operator $A: D(A) \subseteq X \to 2^X$ is accretive if and only if for every $x, y \in D(A), \alpha > 0$ obtains*

$$\|x - y\| \leq \|(x - y) + \alpha(u - v)\|$$

for all $u \in A(x), v \in A(y)$.

Proof. For any $x, y \in D(A)$, there exists $j \in J(x - y)$ such that

$$\langle u - v, j\rangle \geq 0 \text{ for all } u \in A(x), v \in A(y),$$

where $J: X \to 2^{X^*}$ is normalized duality mapping. According to Theorem 4, for every $\alpha > 0$ we have

$$\|x - y\| \leq \|(x - y) + \alpha(u - v)\|$$

for all $u \in A(x), v \in A(y)$.

Conservely, if for every $\alpha > 0$, we have $\|x - y\| \leq \|(x - y) + \alpha(u - v)\|$ for all $u \in A(x), v \in A(y)$, so by Theorem 4 there exists $j \in J(x - y)$ such that $\langle u - v, j\rangle \geq 0$. This means A is an accretive operator. ■

Example 6. An operator $A: \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ defined by

$$A(\bar{x}) = \{(x_2, -x_1)\} \text{ for all } \bar{x} = (x_1, x_2) \in \mathbb{R}^2$$

is accretive because for all $\bar{x} = (x_1, x_2), \bar{y} = (y_1, y_2) \in \mathbb{R}^2$ and $\alpha > 0$ obtain

$$\begin{aligned}
\left\|(\bar{x} - \bar{y}) + \alpha\big(A(\bar{x}) - A(\bar{y})\big)\right\| &= \left\|\big((x_1, x_2) - (y_1, y_2)\big) + \alpha\big((x_2, -x_1) - (y_2, -y_1)\big)\right\| \\
&= \|(x_1 - y_1 + \alpha x_2 - \alpha y_2), (x_2 - y_2 - \alpha x_1 + \alpha y_1)\| \\
&= \left\|\big((x_1 - y_1) + \alpha(x_2 - y_2), (x_2 - y_2) + \alpha(y_1 - x_1)\big)\right\| \\
&= \sqrt{\big((x_1 - y_1) + \alpha(x_2 - y_2)\big)^2 + \big((x_2 - y_2) + \alpha(y_1 - x_1)\big)^2} \\
&= \sqrt{(1 + \alpha^2)(x_1 - y_1)^2 + (1 + \alpha^2)(x_2 - y_2)^2 + \ 2\alpha(x_2 - y_2)(x_1 - y_1 + y_1 - x_1)} \\
&= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \alpha^{2(x_2 - y_2)^2} + \ \alpha^2(y_1 - x_1)^2} \\
&= \sqrt{(1 + \alpha^2)(x_1 - y_1)^2 + (1 + \alpha^2)(x_2 - y_2)^2} \\
&= \sqrt{(1 + \alpha^2)((x_1 - y_1)^2 + (x_2 - y_2)^2)} \\
&= \sqrt{(1 + \alpha^2)}\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \\
&= \sqrt{(1 + \alpha^2)}\|\bar{x} - \bar{y}\| \\
&> \|\bar{x} - \bar{y}\|.
\end{aligned}$$

■

4. *m*-accretive Operators

In this section, we study an accretive operator with some assumptions. Let X be a real normed space and $A: D(A) \subseteq X \to 2^X$ be an accretive operator. For any $\lambda > 0$, we define an operator $I + \lambda A: D(A) \subseteq X \to 2^X$, with I is an identity operator.

Definition 7. *Let X be a real normed space and $A: D(A) \subseteq X \to 2^X$ be an accretive operator. Operator A is said to be m-accretive if $\mathcal{R}(I + \lambda A) = X$, for any $\lambda > 0$.*

Example 8. Operator $A: \mathbb{R}^2 \to 2^{\mathbb{R}^2}$ defined on Example 6 is m-accretive operator. By Example 6, A is accretive. Now, it remains to prove that for any $\lambda > 0$, $\mathcal{R}(I + \lambda A) = \mathbb{R}^2$. For every $\lambda > 0$, it is clear that $\mathcal{R}(I + \lambda A) \subseteq \mathbb{R}^2$. For every $\bar{y} = (y_1, y_2) \in \mathbb{R}^2$ there exists $\bar{x} = (x_1, x_2) \in \mathbb{R}^2$ such that

$$x_1 = \frac{y_1 - \lambda y_2}{1+\lambda^2} \text{ and } x_2 = \frac{\lambda y_1 + y_2}{1+\lambda^2}.$$

Hence, we have

$$\begin{aligned}
(I + \lambda A)(\bar{x}) &= \bar{x} + \lambda A(\bar{x}) \\
&= \left(\frac{y_1 - \lambda y_2}{1+\lambda^2}, \frac{\lambda y_1 + y_2}{1+\lambda^2}\right) + \lambda\left(\frac{\lambda y_1 + y_2}{1+\lambda^2}, \frac{-y_1 + \lambda y_2}{1+\lambda^2}\right) \\
&= \left(\frac{y_1 - \lambda y_2 + \lambda^2 y_1 + \lambda y_2}{1+\lambda^2}, \frac{\lambda y_1 + y_2 - \lambda y_1 + \lambda^2 y_2}{1+\lambda^2}\right) \\
&= (y_1, y_2).
\end{aligned}$$

Therefore $\mathbb{R}^2 \subseteq \mathcal{R}(I + \lambda A)$. Hence, A is m-accretive operator. ■

Now are going to continue this section with some properties of m-accretive operator. These properties will be useful to extend our study in accretive operator and its application in applied mathematics.

Theorem 9. *Let X be a real normed space and $A: D(A) \subseteq X \to 2^X$ be a m-accretive operator. For every $y \in X$, operator $A + y: D(A) \subseteq X \to 2^X$ defined by*

$$(A + y)(x) = A(x) + y,$$

for all $x \in D(A)$, is m-accretive.

Proof. First, we are going to prove for all $y \in X$, operator $A + y$ is accretive. Because A is accretive, for every $x_1, x_2 \in D(A)$ there exists $j \in J(x_1 - x_2)$ such that $\langle u - v, j\rangle \geq 0$, for all $u \in A(x_1), v \in A(x_2)$. Hence $\langle u + y - (v + y), j\rangle \geq 0$, for all $u + y \in A(x_1) + y, v + y \in A(x_2) + y$. Therefore $A + y$ is accretive.

Second, we are going to prove that $\mathcal{R}\big(I + \lambda(A + y)\big) = X$, for all $\lambda > 0$. Given any $\lambda > 0$ and $z \in X$. Because A is m-accretive, there exists $x \in D(A)$ such that $z - \lambda y \in (I + \lambda A)(x)$. Hence, we have

$$z \in x + \lambda A(x) + \lambda y = \big(I + \lambda(A + y)\big)(x).$$

This means, $z \in \mathcal{R}(I + \lambda(A + y))$, so $A + y$ is m-accretive. ■

Before we continue this section, we need to recall a closed multivalued operator. Let X, Y be real normed spaces. Operator $A: D(A) \subseteq X \to 2^X$ is closed if for any convergent sequence $(x_n) \subset D(A)$ and for any convergent sequence (y_n), where $y_n \in A(x_n)$ for all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} x_n \in D(A) \text{ and } \lim_{n\to\infty} y_n \in A\left(\lim_{n\to\infty} x_n\right).$$

Theorem 10. *Let X be a real normed space. If operator $A: D(A) \subseteq X \to 2^X$ is m-accretive then A is closed.*

Proof. Let $(x_n) \subset D(A)$ be a convergent sequence and$x_0 \in X$ be its limit. Let (y_n) be convergent sequence, where $y_n \in A(x_n)$ for all $n \in \mathbb{N}$, and $y_0 \in X$ be its limit. By Corollary 5, for every $n \in \mathbb{N}, x \in D(A)$ and $\alpha > 0$ we have

$$\|x_n - x\| \le \|(x_n - x) + \alpha(y_n - y)\|, \text{for all } y \in A(x).$$

Therefore, if $n \to \infty$ then for every $\alpha > 0$ we have

$$\|x_0 - x\| \le \|(x_0 - x) + \alpha(y_0 - y)\|$$

for all $x \in D(A), y \in A(x)$. By Theorem 4, there exists $j \in J(x_0 - x)$ such that $\langle y_0 - y, j\rangle \ge 0$. Hence

$$\langle y - y_0, j\rangle \le 0. \tag{6}$$

Because A is m-accretive then $x_0 + y_0 \in \mathcal{R}(I + A)$. There exists $x_* \in D(A)$ such that $x_0 + y_0 \in (I + A)(x_*)$. This means that there exists $y_* \in A(x_*)$ such that $x_0 + y_0 = x_* + y_*$. By Eq. (6), we have $\langle x_0 - x_*, j\rangle = \langle y_* - y_0, j\rangle \le 0$. Hence $j \in J(x_0 - x_*)$ and consequently this result satisfies

$$\|x_0 - x_*\|^2 = \langle x_0 - x_*, j\rangle \le 0,$$

so $\|x_0 - x_*\| = 0$. This means $x_0 = x_*$ and $y_0 = y_*$. Therefore, $x_0 \in D(A)$ and $y_0 \in A(x_0)$. Finally we have

$$y_0 \in A\left(\lim_{n\to\infty} x_n\right).$$

■

By Theorem 10, we derive the following result.

Theorem 11. *Let X be a real normed space and $B: X \to X$ be a continuous operator. If $A: D(A) \subseteq X \to 2^X$ be a m-accretive operator, then A+B is closed.*

Proof. Let $(x_n) \subset D(A)$ be any convergent sequence and $x_* \in X$ be its limit. Let (y_n) be any convergent sequence, with $y_n \in (A + B)(x_n)$ for all $n \in \mathbb{N}$. There exists $v_n \in A(x_n)$ such that $y_n = v_n + B(x_n)$. Because A is m-accretive, by Theorem 10 we have A is closed. Thus $x_* \in D(A)$ and

$$\lim_{n\to\infty} v_n \in A(x_*). \tag{7}$$

Hence we have

$$\begin{aligned}\lim_{n\to\infty} y_n &= \lim_{n\to\infty}\big(v_n + B(x_n)\big)\\ &= \lim_{n\to\infty} v_n + \lim_{n\to\infty} B(x_n)\\ &= \lim_{n\to\infty} v_n + B\left(\lim_{n\to\infty} x_n\right)\\ &= \lim_{n\to\infty} v_n + B(x_*)\end{aligned}$$

Because of Eq. (7), then

$$\lim_{n\to\infty} y_n \in A(x_*) + B(x_*) = (A + B)(x_*).$$

This means $A + B$ is closed. ■

5. Conclusion

Let X be a real normed space. An operator $A: D(A) \subseteq X \to 2^X$ is accretive if and only if for every $x, y \in D(A), \alpha > 0$ obtains

$$\|x - y\| \le \|(x - y) + \alpha(u - v)\|$$

for all $u \in A(x), v \in A(y)$. If $\mathcal{R}(I + \lambda A) = X$, for any $\lambda > 0$, then operator A is said to be m-accretive. For an operator m-accretive A, operator A is closed and for every $y \in X$, operator $A + y: D(A) \subseteq X \to 2^X$ defined by

$$(A + y)(x) = A(x) + y,$$

for all $x \in D(A)$, is m-accretive. Furthermore, if $B: X \to X$ be a continuous operator, then A+B is closed.

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